

CONTACT KÄHLER MANIFOLDS: SYMMETRIES AND DEFORMATIONS

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ABSTRACT. We study complex compact Kähler manifolds X carrying a contact structure. If X is almost homogeneous and $b_2(X) \geq 2$, then X is a projectivised tangent bundle (this was known in the projective case even without assumption on the existence of vector fields). We further show that a global projective deformation of the projectivised tangent bundle over a projective space is again of this type unless it is the projectivisation of a special unstable bundle over a projective space. Examples for these bundles are given in any dimension.

1. INTRODUCTION

A contact structure on a complex manifold X is in some sense the opposite of a foliation: there is a vector bundle sequence

$$0 \rightarrow F \rightarrow T_X \rightarrow L \rightarrow 0,$$

where T_X is the tangent bundle and L a line bundle, with the additional property that the induced map

$$\bigwedge^2 F \rightarrow L, \quad v \wedge w \mapsto [v, w]/F$$

is everywhere non-degenerate.

Suppose now that X is compact and Kähler or projective. If $b_2(X) = 1$, then at least conjecturally the structure is well-understood: X should arise as minimal orbit in the projectivised Lie algebra of contact automorphisms. Beauville [Be98] proved this conjecture under the additional assumption that the group of contact automorphisms is reductive and that the contact line bundle L has “enough” sections.

If $b_2(X) \geq 2$ and X is projective, then, due to [KPSW00] and [De02], X is a projectivized tangent bundle $\mathbb{P}(T_Y)$ (in the sense of Grothendieck, taking hyperplanes) over a projective manifold Y (and conversely every such projectivised tangent bundle carries a contact structure). If X is only Kähler, the analogous conclusion is unknown. By [De02], the canonical bundle K_X is still not pseudo-effective in the Kähler setting, but—unlike in the projective case—it is not known whether this implies uniruledness of X .

If however X has enough symmetries, then we are able to deal with this situation:

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Theorem 1.1. *Let X be an almost homogeneous compact Kähler manifold carrying a contact structure. If $b_2(X) \geq 2$, then there is a compact Kähler manifold Y such that $X \simeq \mathbb{P}(T_Y)$.*

Here a manifold is said to be almost homogeneous, if the group of holomorphic automorphisms acts with an open orbit. Equivalently, the holomorphic vector fields generate the tangent bundle T_X at some (hence at the general) point.

In this setting it might be interesting to try to classify all compact almost homogeneous Kähler manifolds X of the form $X = \mathbb{P}(T_Y)$. Section 4 studies this question in dimension 3.

In the second part of the paper we treat the deformation problem for projective contact manifolds. We consider a family

$$\pi: \mathcal{X} \rightarrow \Delta$$

of projective manifolds over the 1-dimensional disc $\Delta \subset \mathbb{C}$. Suppose that all $X_t = \pi^{-1}(t)$ are contact for $t \neq 0$. Is then X_0 also a contact manifold?

Suppose first that $b_2(X_t) = 1$. Then—as discussed above— X_t should be homogeneous for $t \neq 0$. Assuming homogeneity, the situation is well-understood by the work of Hwang and Mok. In fact, then X_0 is again homogeneous with one surprising 7-dimensional exception, discovered by Pasquier-Perrin [PP10] and elaborated further by Hwang [Hw10]. Therefore one has rigidity and the contact structure survives unless the Pasquier-Perrin case happens, where the contact structure does not survive. We refer to [Hw10] and the references given at the beginning of section 5. Therefore—up to the homogeneity conjecture—the situation is well-understood.

If $b_2(X_t) \geq 2$, the situation gets even more difficult; so we will assume that X_t is homogeneous for $t \neq 0$. We give a short argument in sect. 2, showing that then X_t is either $\mathbb{P}(T_{\mathbb{P}_n})$ or a product of a torus and \mathbb{P}_n . Then we investigate the *global projective rigidity* of $\mathbb{P}(T_{\mathbb{P}_n})$:

Theorem 1.2. *Let $\pi: \mathcal{X} \rightarrow \Delta$ be a projective family of compact manifolds. If $X_t \simeq \mathbb{P}(T_{\mathbb{P}_n})$ for $t \neq 0$, then either $X_0 \simeq \mathbb{P}(T_{\mathbb{P}_n})$ or $X_0 \simeq \mathbb{P}(V)$ with some unstable vector bundle V on \mathbb{P}_n .*

The assumption that X_0 is projective is indispensable for our proof. In general, X_0 is only Moishezon, and in particular methods from Mori theory fail. In case X_0 is even assumed to be Fano, the theorem was proved by Wiśniewski [Wi91a]; in this case $X_0 \simeq \mathbb{P}(T_{\mathbb{P}_n})$. The case $X_0 \simeq \mathbb{P}(V)$ with an unstable bundle really occurs; we provide examples in all dimensions in section 6. In this case X_0 is no longer a contact manifold.

In general, without homogeneity assumption, X_t is the projectivisation of the tangent bundle of some projective variety Y_t ; here we have only some

partial results, see Proposition 5.8. If however X_t is again homogeneous ($t \neq 0$) and not the projectivization of the tangent bundle of a projective space, then X_t is a product of a torus A_t and a projective space, and we obtain a rather clear picture, described in Section 7.

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2. HOMOGENEOUS KÄHLER CONTACT MANIFOLDS

We first study homogeneous manifolds which are projectivized tangent bundles.

Proposition 2.1. *Let Y be compact Kähler. Then $X = \mathbb{P}(T_Y)$ is homogeneous if and only if Y is a torus or $Y = \mathbb{P}_n$.*

Proof. One direction being clear, assume that X is homogeneous; thus Y is homogeneous, too. The theorem of Borel-Remmert [BR62] says that

$$Y \cong A \times G/P$$

where G/P is a rational homogeneous manifold (G a semi-simple complex Lie group and P a parabolic subgroup) and A a torus, one factor possibly of dimension 0. Let $d = \dim A \geq 0$.

We first assume that $d > 0$. Then $T_Y = \mathcal{O}_Y^d \oplus T_{G/P}$ leading to an inclusion

$$Z := \mathbb{P}(\mathcal{O}_Y^d) \subset X$$

with normal bundle

$$N_{Z/X} = \mathcal{O}_Z(1) \otimes \pi^* q^*(\Omega_{G/P}^1) = p^*(\mathcal{O}(1)) \otimes \pi^* q^*(\Omega_{G/P}^1).$$

Here $\pi: X \rightarrow Y$, $p: Z = \mathbb{P}_{d-1} \times Y \rightarrow \mathbb{P}_{d-1}$ and $q: Y \rightarrow G/P$ are the projections. Now, X being homogeneous, $N_{Z/X}$ is spanned. This is only possible when $\dim G/P = 0$ so that $Y = A$.

If $d = 0$, then X is rational homogeneous, hence Fano. This is to say that T_Y is ample, hence $Y = \mathbb{P}_n$ (we do not need Mori's theorem here because Y is already homogeneous). \square

Proposition 2.1 is now applied to obtain

Proposition 2.2. *Let X be a homogeneous compact Kähler manifold with contact structure and $\dim X = 2n - 1$. Then either X is a Fano manifold (and therefore $X \simeq \mathbb{P}(T_{\mathbb{P}_n})$, by Prop. 2.1, unless $b_2(X) = 1$) or*

$$X \cong A \times \mathbb{P}_{n-1} = \mathbb{P}(T_A),$$

where A denotes a complex torus of dimension n and T_A its holomorphic tangent bundle.

Proof. Again by the theorem of Borel-Remmert, $X \cong A \times G/P$ where G/P is rational-homogeneous and A a torus, one factor possibly of dimension 0. If A does not appear, then X is Fano with $b_2(X) \geq 2$ and therefore by [KPSW00] of the form $X = \mathbb{P}(T_Y)$. Then we conclude by Prop. 2.1.

So we may assume $\dim A > 0$. Since a torus does not admit a contact structure, it follows that the factor G/P is nontrivial, i.e. $\dim G/P \geq 1$. We consider the projection $\pi: X \cong A \times G/P \rightarrow A$. Every fiber is G/P and in particular a Fano manifold. We may therefore use the arguments of [KPSW00], Proposition 2.11, to conclude that every fiber is \mathbb{P}_{n-1} . Note that the arguments used in [KPSW00], Proposition 2.11 do not use the assumption that X is projective. This completes the proof. \square

3. THE ALMOST HOMOGENEOUS CASE

The aim of this section is to generalize the previous section to almost homogeneous contact manifolds.

3.1. Almost homogeneous projectivized tangent bundles. We begin with the following general observation.

Lemma 3.1. *Let Y be a compact complex manifold and let $X = \mathbb{P}(T_Y)$ be its projectivised tangent bundle. If X is almost homogeneous, then Y is almost homogeneous.*

We already mentioned that if X is homogeneous, so is Y .

Proof. Let $\pi: X \rightarrow Y$ be the bundle projection and consider the relative tangent sequence

$$0 \rightarrow T_{X/Y} \rightarrow T_X \rightarrow \pi^*T_Y \rightarrow 0.$$

Since at a general point of X the tangent bundle T_X is spanned by global sections, so is π^*T_Y . So if $y \in Y$ is general, if $x \in \pi^{-1}(y)$ is general and $v \in (\pi^*T_Y)_x$, then there exists

$$s \in H^0(X, \pi^*(T_Y))$$

such that $s(x) = v$. Since $s = \pi^*(t)$ with $t \in H^0(Y, T_Y)$, we obtain $t(y) = v \in T_{Y,y}$. Thus Y is almost homogeneous. \square

Remark 3.2. Note that, conversely, the projectivized tangent bundle $X = \mathbb{P}(T_Y)$ of an almost homogeneous manifold Y is in general **not** almost homogeneous. This is illustrated by the following examples.

Example 3.3. We start in a quite general setting with a projective manifold Y of dimension n . We assume that Y is almost homogeneous with $h^0(Y, T_Y) = n$. Furthermore we assume

$$h^0(Y, \Omega_Y^1 \otimes T_Y) = h^0(Y, \text{End}(T_Y)) = 1, \quad (1)$$

an assumption which is e.g. satisfied if T_Y is stable for some polarization. We let $X = \mathbb{P}(T_Y)$ be the projectivized tangent bundle with projection $\pi: X = \mathbb{P}(T_Y) \rightarrow Y$ and hyperplane bundle $\mathcal{O}_X(1)$. Pushing forward the relative Euler sequence to Y yields

$$0 \rightarrow \mathcal{O}_Y \rightarrow \Omega_Y^1 \otimes \pi_*(\mathcal{O}_X(1)) \rightarrow \pi_*T_{X/Y} \rightarrow 0.$$

Since $\pi_*(\mathcal{O}_X(1)) = T_Y$, we obtain

$$0 \rightarrow \mathcal{O}_Y \rightarrow \Omega_Y^1 \otimes T_Y \rightarrow \pi_*T_{X/Y} \rightarrow 0.$$

This sequence splits via the trace map $\Omega_Y^1 \otimes T_Y \simeq \text{End}(T_Y) \rightarrow \mathcal{O}_Y$, so we obtain the exact sequence

$$0 \rightarrow H^0(Y, \mathcal{O}_Y) \rightarrow H^0(Y, \Omega_Y^1 \otimes T_Y) \rightarrow H^0(Y, \pi_*T_{X/Y}) \rightarrow 0.$$

Using assumption (1) we find

$$H^0(X, T_{X/Y}) = H^0(Y, \pi_*T_{X/Y}) = 0.$$

Now the relative tangent sequence with respect to $\pi: X \rightarrow Y$ yields an exact sequence

$$0 \rightarrow H^0(X, T_{X/Y}) \rightarrow H^0(X, T_X) \rightarrow H^0(X, \pi^*(T_Y)) \simeq H^0(Y, T_Y)$$

and therefore

$$h^0(T_X) \leq h^0(T_Y).$$

Hence $h^0(T_X) \leq n$, and X cannot be almost homogeneous.

Notice that an inequality $h^0(T_X) \leq 2n - 2$ suffices to conclude that X is not almost homogeneous. Therefore we could weaken the assumptions $h^0(T_Y) = n$ and $h^0(\text{End}(T_Y)) = 1$ to

$$h^0(T_Y) + h^0(\text{End}(T_Y)) \leq 2n - 2.$$

We give two specific examples.

First, let Y be a del Pezzo surface of degree six, i.e., a three-point blow-up of \mathbb{P}_2 . Its automorphisms group is $(\mathbb{C}^*)^2 \rtimes S_3$. In particular, Y is almost homogeneous and $h^0(T_Y) = 2$. Since $h^0(\text{End}(T_{\mathbb{P}_2})) = 1$ and Y is a blow up of \mathbb{P}_2 , each endomorphism of T_Y induces an endomorphism of $T_{\mathbb{P}_2}$ and it follows that

$$h^0(T_Y \otimes \Omega_Y^1) = h^0(\text{End}(T_Y)) = 1. \quad (2)$$

Hence the assumptions of our previous considerations are fulfilled and $X = \mathbb{P}(T_Y)$ is not almost homogeneous.

Here is an example with $b_2(Y) = 1$. We let Y be the Mukai-Umemura Fano threefold of type V_{22} , [MU83]. Here $h^0(T_Y) = 3$ and Y is almost homogeneous with $\text{Aut}^0(Y) = \text{SL}_2(\mathbb{C})$. Since T_Y is known to be stable (see e.g. [PW95]), again all assumptions are satisfied and $X = \mathbb{P}(T_Y)$ is not almost homogeneous.

3.2. The Albanese map for almost homogeneous manifolds. A well-known theorem of Barth-Oeljeklaus determines the structure of the Albanese map of an almost homogeneous Kähler manifold.

Theorem 3.4 ([BO74]). *Let X be an almost homogeneous compact Kähler manifold. Then the Albanese map $\alpha: X \rightarrow A$ is a fiber bundle. The fibers are connected, simply-connected and projective.*

Remark 3.5. *The fibers X_a of α are almost homogeneous.*

Proof. Let $x, y \in X_a$ be two general points. Then there exists $f \in \text{Aut}(X)$ with $f(x) = y$. Since the automorphism f is fiber preserving, we obtain an automorphism of X_a mapping x to y . \square

3.3. The case $q(X) = 0$. If the irregularity of X is $q(X) = 0$, the Albanese map is trivial, and it follows that X itself is simply-connected and projective.

Lemma 3.6. *Let X be an almost homogeneous compact Kähler manifold with contact structure. If $q(X) = 0$ and $b_2(X) \geq 2$, then $X \cong \mathbb{P}(T_Y)$ is a projectivised tangent bundle.*

Proof. X being projective, the results of [KPSW00] apply. Combining them with [De02] (cf. Corollary 4) yields the desired result. \square

Remark 3.7. The case where $q(X) = 0$ and $b_2(X) = 1$ remains to be studied. Here X is an almost homogeneous Fano manifold. It would be interesting to find out whether the results of [Be98] apply. I.e., one has to check whether $\text{Aut}(X)$ is reductive and whether the map associated with the contact line bundle L is generically finite.

In order to study the second property, consider the long exact sequence

$$0 \rightarrow H^0(X, F) \rightarrow H^0(X, T_X) \rightarrow H^0(X, L) \rightarrow \dots$$

If $H^0(X, F) \neq 0$ then X has more than one contact structure [Le95], Prop.2.2, hence Corollary 4.5 of [Ke01] implies that $X \cong \mathbb{P}_{2n+1}$ or $X \cong \mathbb{P}(T_Y)$.

If $H^0(X, F) = 0$ then L has “many sections” and the map associated with L is expected to be generically finite.

3.4. The case $q(X) \geq 1$. If the irregularity of X is positive, then the Albanese map $\alpha: X \rightarrow A$ is a fiber bundle. We denote its fiber by X_a .

Lemma 3.8. *Let X be an almost homogeneous compact Kähler manifold with contact structure and $q(X) \geq 1$. If the fiber X_a of the Albanese map fulfills $b_2(X_a) = 1$, then $X \cong \mathbb{P}(T_A) = \mathbb{P}_n \times A$, where A is the Albanese torus of X .*

Proof. Since $b_2(X_a) = 1$, then X_a (being uniruled) is a Fano manifold. We may therefore apply Proposition 2.11 of [KPSW00] (which works perfectly in our situation) to conclude that $\alpha: X \rightarrow A$ is a \mathbb{P}_n -bundle. The proof of Theorem 2.12 in [KPSW00] can now be adapted to conclude that $X \cong \mathbb{P}(T_A)$. To be more specific, we already know in our situation that $X = \mathbb{P}(\mathcal{E})$ with $\mathcal{E} = \alpha_*(L)$. The only thing to be verified is the isomorphism $\mathcal{E} \simeq T_A$. But this is seen as in the last part of the proof of Theorem 2.12 in [KPSW00], since section 2.1 of [KPSW00] works on any manifold.

So $X \simeq \mathbb{P}(T_A)$ and $X \cong \mathbb{P}_n \times A$. \square

It remains to study the case where the fiber X_a fulfills $b_2(X_a) \geq 2$. In this case we consider a relative Mori contraction (over A ; the projection is a projective morphism, [Na87], (4.12))

$$\varphi: X \rightarrow Y.$$

Lemma 3.9. *We have $\dim X > \dim Y$.*

Proof. The lemma follows from the fact that the restriction map $\varphi_a = \varphi|_{X_a}$ is not birational. This can be shown by the same arguments as in Lemma 2.10 of [KPSW00] using the length of the contraction and the restriction of the contact line bundle to the fiber X_a . Again the projectivity of X is not needed in Lemma 2.10. \square

As above, we may now apply Proposition 2.11 of [KPSW00] and conclude that the general fiber of φ is \mathbb{P}_n . It remains to check that φ is a \mathbb{P}_n -bundle and $X \cong \mathbb{P}(T_Y)$. This is done again as in Theorem 2.12 of [KPSW00] with Fujita's result generalized to the Kähler setting by Lemma 3.10. Also the compactness assumption in [Fu85] is not necessary, this will be important later.

Lemma 3.10. *Let X be a complex manifold, $f: X \rightarrow S$ a proper surjective map to a normal complex space S . Let L be a relatively ample line bundle on X such that $(F, L_F) \simeq (\mathbb{P}_r, \mathcal{O}(1))$ for a general fiber F of f . If f is equidimensional, then f is a \mathbb{P}_r -bundle.*

In total, we obtain

Theorem 3.11. *Let X be a compact almost homogeneous Kähler contact manifold, $b_2(X) \geq 2$. Then $X = \mathbb{P}(T_Y)$ with a compact Kähler manifold Y .*

The arguments above actually also show the following.

Theorem 3.12. *Let X be a compact Kähler contact manifold. Let $\phi: X \rightarrow Y$ be a surjective map with connected fibers such that $-K_X$ is ϕ -ample and such that $\rho(X/Y) = 1$ (we do not require the normal variety Y to be Kähler). Then Y is smooth and $X = \mathbb{P}(T_Y)$.*

One might wonder whether this is still true when X is Moishezon or bimeromorphic to a Kähler manifold. Although there is no apparent reason why the theorem should not hold in this context, at least the methods of proof completely fail. More generally, also the assumption that X is almost homogeneous should be unnecessary. If X is still Kähler, a Mori theory in the non-algebraic case seems unavoidable. Already the question whether X is uniruled is hard.

3.5. Conclusion and open questions. (1) In all but one case we find that a compact almost homogeneous Kähler contact manifold X has the structure of a projectivised tangent bundle. The remaining case where $q(X) = 0$ and $b_2(X) = 1$ is discussed in Remark 3.7.

(2) Can one classify all Y (necessarily almost homogeneous) such that $\mathbb{P}(T_Y)$ is almost homogeneous? The case where $\dim Y = 2$ will be treated in the next section. One might also expect that if $Y = G/P$, then X should be almost homogeneous. In case Y is a Grassmannian or a quadric, this has been checked by Goldstein [Go83]. Of course, if $Y = \mathbb{P}_n$, then X is even homogeneous.

4. ALMOST HOMOGENEOUS CONTACT THREEFOLDS

In this section we specialize to almost homogeneous contact manifolds in dimension 3.

Theorem 4.1. *Let X be a smooth compact Kähler threefold which is of the form $X = \mathbb{P}(T_Y)$ for some compact (Kähler) surface Y .*

- (1) *If X is almost homogeneous, then Y is a minimal surface or a blow-up of \mathbb{P}_2 or $Y = \mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-n))$ for some $n \geq 0$, $n \neq 1$*
- (2) *If Y is minimal, then X is almost homogeneous if and only if Y is one of the following surfaces.*
 - $Y = \mathbb{P}_2$
 - $Y = \mathbb{F}_n$ for some $n \geq 0$, $n \neq 1$
 - Y is a torus
 - $Y = \mathbb{P}(\mathcal{E})$ with \mathcal{E} a vector bundle of rank 2 over an elliptic curve which is either a direct sum of two topologically trivial line bundles or the non-split extension of two trivial line bundles.

Proof. Suppose X is almost homogeneous. Then Y is almost homogeneous, too (Lemma 3.1). By Potters' classification [Po68], Y is one of the following.

- (1) $Y = \mathbb{P}_2$
- (2) $Y = \mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-n))$ for some $n \geq 0$, $n \neq 1$
- (3) Y is a torus

- (4) $Y = \mathbb{P}(\mathcal{E})$ with \mathcal{E} a vector bundle of rank 2 over an elliptic curve which is either a direct sum of two topologically trivial line bundles or the non-split extension of two trivial line bundles
- (5) Y is a certain blow-up of \mathbb{P}_2 or of \mathbb{F}_n .

This already shows the first claim of the theorem, and it suffices to assume Y to be a minimal surface of the list and to check whether $X = \mathbb{P}(T_Y)$ is almost homogeneous. In cases (1) and (3) this is clear; X is even homogeneous.

To proceed further, consider the tangent bundle sequence

$$0 \rightarrow T_{X/Y} \rightarrow T_X \rightarrow \pi^*(T_Y) \rightarrow 0.$$

Notice

$$h^0(T_{X/Y}) = h^0(-K_{X/Y}) = h^0(S^2 T_Y \otimes K_Y).$$

Applying π_* and observing that the connecting morphism

$$T_Y \rightarrow R^1 \pi_*(T_{X/Y})$$

(induced by the Kodaira-Spencer maps) vanishes since π is locally trivial, it follows that

$$H^0(X, T_X) \rightarrow H^0(X, \pi^*(T_Y)) = H^0(Y, T_Y)$$

is surjective. It therefore

$$H^0(X, T_{X/Y}) \simeq H^0(Y, S^2 T_Y \otimes K_Y) \neq 0, \quad (*)$$

the tangent bundle T_X is obviously spanned and therefore X is almost homogeneous.

In case (4), $(*)$ is now easily verified: Let $p: \mathbb{P}(\mathcal{E}) \rightarrow C$ be the \mathbb{P}_1 -fibration over the elliptic curve C . The tangent bundle sequence reads

$$0 \rightarrow -K_Y \rightarrow T_Y \rightarrow \mathcal{O}_Y \rightarrow 0.$$

Since T_Y is generically spanned, the map $H^0(\mathcal{O}_Y) \rightarrow H^1(-K_Y)$ must vanish, so that the sequence splits:

$$T_Y \simeq -K_Y \oplus \mathcal{O}_Y.$$

Thus $S^2 T_Y \otimes K_Y \simeq -K_Y \oplus \mathcal{O}_Y \oplus K_Y$ and $(*)$ follows.

Now if $Y = \mathbb{F}_n$ as in (2), let $p: Y \rightarrow \mathbb{P}_1$ be the natural projection. The relative tangent sequence then reads

$$0 \rightarrow T_{Y/\mathbb{P}_1} \rightarrow T_Y \rightarrow p^* \mathcal{O}_{\mathbb{P}_1}(2) \rightarrow 0. \quad (**)$$

Taking the second symmetric power and tensorizing with K_Y yields

$$0 \rightarrow T_Y \otimes T_{Y/\mathbb{P}_1} \otimes K_Y \rightarrow S^2 T_Y \otimes K_Y \rightarrow p^* \mathcal{O}_{\mathbb{P}_1}(4) \otimes K_Y \rightarrow 0,$$

so, by $(**)$, we obtain an inclusion

$$H^0(T_{Y/\mathbb{P}_1}^{\otimes 2} \otimes K_Y) \subset H^0(S^2 T_Y \otimes K_Y).$$

Now by the relative Euler sequence, $T_{Y/\mathbb{P}_1} \simeq \mathcal{O}_Y(2) \otimes p^* \mathcal{O}_{\mathbb{P}_1}(n)$, and thus

$$H^0(T_{Y/\mathbb{P}_1}^{\otimes 2} \otimes K_Y) \simeq H^0(\mathcal{O}_Y(2) \otimes p^* \mathcal{O}_{\mathbb{P}_1}(n-2)).$$

Now since

$$p_*(\mathcal{O}_Y(2) \otimes p^* \mathcal{O}_{\mathbb{P}_1}(n-2)) \simeq \mathcal{O}_{\mathbb{P}_1}(n-2) \oplus \mathcal{O}_{\mathbb{P}_1}(-2) \oplus \mathcal{O}_{\mathbb{P}_1}(-n-2),$$

we have shown (*) to be true for $n \geq 2$. If $n = 0$, i.e., $Y \simeq \mathbb{P}_1 \times \mathbb{P}_1$, the sequence (**) splits and an easy calculation shows that (*) is satisfied also in this case. \square

Remark 4.2. The case that Y is a non-minimal rational surface in Theorem 4.1 could be further studied, but this is a rather tedious task.

5. DEFORMATIONS I: THE RATIONAL CASE

We consider a family $\pi: \mathcal{X} \rightarrow \Delta$ of compact manifolds over the unit disc $\Delta \subset \mathbb{C}$. As usual, we let $X_t = \pi^{-1}(t)$. We shall assume X_t to be a projective manifold for all t , so we are only interested in projective families here. If now X_t is a contact manifold for $t \neq 0$, when is X_0 still a contact manifold? If $b_2(X_t) = 1$, there is a counterexample due to [PP10], see also [Hw10]. Here the X_t are 7-dimensional rational-homogeneous contact manifolds and X_0 is a non-homogeneous non-contact manifold. If one believes that any Fano contact manifold with $b_2 = 1$ is rational-homogeneous, then due to the results of Hwang and Mok, this is the only example where a limit of contact manifolds with $b_2 = 1$ is not contact.

If $b_2(X_t) \geq 2$, it is no longer true that the limit X_0 is always a contact manifold, as can be seen from the following example: We let $\mathcal{Y} \rightarrow \Delta$ be a family of compact manifolds such that $Y_t \simeq \mathbb{P}_1 \times \mathbb{P}_1$ for $t \neq 0$ and $Y_0 \simeq \mathbb{F}_2$. Then there exist line bundles \mathcal{L}_1 and \mathcal{L}_2 on \mathcal{Y} such that $\mathcal{L}_1|_{Y_t} \simeq \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_1}(2, 0)$ and $\mathcal{L}_2|_{Y_t} \simeq \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_1}(0, 2)$ for every $t \neq 0$. If we let $\mathcal{X} := \mathbb{P}(\mathcal{L}_1 \oplus \mathcal{L}_2)$, then $X_t \simeq \mathbb{P}(T_{Y_t})$ for $t \neq 0$, but $X_0 \not\simeq \mathbb{P}(T_{Y_0})$.

However $\mathbb{P}(T_{\mathbb{P}_1 \times \mathbb{P}_1})$ is not homogeneous; in fact by Proposition 2.1, $\mathbb{P}(T_{\mathbb{P}_n})$ is the only homogeneous rational contact manifold with $b_2 \geq 2$. In this prominent case we prove global projective rigidity, i.e., $X_0 = \mathbb{P}(T_{\mathbb{P}_n})$, unless X_0 is the projectivization of some unstable bundle, so that both contact structures survive in the limit. In the “unstable case”, the contact structure does not survive. The special case where X_0 is Fano is due to Wiśniewski [Wi91a]; here global rigidity always holds.

There is a slightly different point of view, asking whether projective limits of rational-homogeneous manifolds are again rational-homogeneous. As before, if $b_2(X_t) = 1$, this is true by the results of Hwang and Mok with the 7-dimensional exception. In case $b_2(X_t) \geq 2$, this is false in general (e.g. for $\mathbb{P}_1 \times \mathbb{P}_1$), but the picture under which circumstances global rigidity is still true is completely open.

Theorem 5.1. *Let $\pi: \mathcal{X} \rightarrow \Delta$ be a family of compact manifolds. Assume $X_t \simeq \mathbb{P}(T_{\mathbb{P}_n})$ for $t \neq 0$. If X_0 is projective, then either $X_0 \simeq \mathbb{P}(T_{\mathbb{P}_n})$ or $X_0 \simeq \mathbb{P}(V)$ with some unstable vector bundle V on \mathbb{P}_n .*

Proof. Since $K_{\mathcal{X}}$ is not π -nef, there exists a relative Mori contraction (see [Na87], (4.12), we may shrink Δ)

$$\Phi: \mathcal{X} \rightarrow \mathcal{Y}$$

over Δ . Put $\Delta^* = \Delta \setminus \{0\}$ and $\mathcal{X}^* = \mathcal{X} \setminus X_0$; $\mathcal{Y}^* = \mathcal{Y} \setminus Y_0$. Now $\phi_t = \Phi|_{X_t}$ is a Mori contraction for any t (cp. [KM92], (12.3.4), but this is pretty clear in our simple situation), *unless possibly ϕ_t is biholomorphic for $t \neq 0$.*

Now since \mathcal{X} , Δ and π are smooth, the latter case cannot occur by [Wi91b], (1.3), so ϕ_t is the contraction of an extremal ray for any $t \in \Delta$. Let $\tau: \mathcal{Y} \rightarrow \Delta$ be the induced projection and set $Y_t = \tau^{-1}(t)$, so that $Y_t \simeq \mathbb{P}_n$ for $t \neq 0$. Since \mathcal{Y} is normal, the normal variety Y_0 must also have dimension n .

From the exponential sequence, Hodge decomposition and the topological triviality of the family \mathcal{X} , it follows that

$$\text{Pic}(\mathcal{X}) \simeq H^2(\mathcal{X}, \mathbb{Z}) \simeq \mathbb{Z}^2$$

and that

$$\text{Pic}(X_0) \simeq H^2(X_0, \mathbb{Z}^2) \simeq \mathbb{Z}^2.$$

Furthermore, the restriction $\text{Pic}(\mathcal{X}) \rightarrow \text{Pic}(X_0)$ is bijective. As an immediate consequence, we can write

$$-K_{\mathcal{X}} = n\mathcal{H}$$

with a line bundle \mathcal{H} on \mathcal{X} . Let $\mathcal{H}_t = \mathcal{H}|_{X_t}$ so that $\mathcal{H}_t \simeq \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}_n})}(1)$ for $t \neq 0$.

Claim 5.2. $Y_0 \simeq \mathbb{P}_n$.

In fact, by our previous considerations, there is a unique line bundle \mathcal{L} on \mathcal{X} such that

$$\mathcal{L}|_{X_t} = \phi_t^*(\mathcal{O}_{\mathbb{P}_n}(1))$$

for $t \neq 0$. Moreover $\mathcal{L}|_{X_0} = \phi_0^*(\mathcal{L}')$ with some ample line bundle \mathcal{L}' on Y_0 . Therefore by semi-continuity,

$$h^0(\mathcal{L}') = h^0(\mathcal{L}|_{X_0}) \geq n + 1$$

and

$$c_1(\mathcal{L}')^n = 1.$$

Hence by results of Fujita [Fu90], (I.1.1), see also [BS95], (III.3.1), we have $(Y_0, \mathcal{L}') \simeq (\mathbb{P}_n, \mathcal{O}(1))$.

In particular we obtain

Sub-Corollary 5.3. \mathcal{Y} is smooth and $\mathcal{Y} \simeq \mathbb{P}_n \times \Delta$.

Next we notice that the general fiber of ϕ_0 must be \mathbb{P}_{n-1} , since it is a smooth degeneration of fibers of ϕ_t (by the classical theorem of Hirzebruch–Kodaira).

One main difficulty is that ϕ_0 might not be equidimensional. If we know equidimensionality, we may apply [Fu85, 2.12] to conclude that $X_0 = \mathbb{P}(\mathcal{E}_0)$ with a locally free sheaf \mathcal{E}_0 on Y_0 .

We introduce the torsion free sheaf

$$\mathcal{F} = \Phi_*(\mathcal{H}) \otimes \mathcal{O}_Y(-1).$$

Since

$$\text{codim } \Phi^{-1}(\text{Sing}(\mathcal{F})) \geq 2,$$

the sheaf \mathcal{F} is actually reflexive and of course locally free outside Y_0 . In the following Sublemma we will prove that \mathcal{F} is actually locally free.

Sub-Lemma 5.4. *\mathcal{F} is locally free and therefore $\mathcal{X} = \mathbb{P}(\mathcal{F})$.*

Proof. As explained above, it is sufficient to show that

$$\phi_0: X_0 \rightarrow \mathbb{P}_n$$

is equidimensional. So let F_0 be an irreducible component of a fiber of ϕ_0 . Then F_0 gives rise to a class

$$[F_0] \in H^{2k}(X_0, \mathbb{Q}),$$

where we denote by k the codimension of F_0 in X . Obviously $k \leq n$, and we must exclude the case that $k < n$.

So we assume in the following that $k < n$. Then, since X_0 is homeomorphic to $\mathbb{P}(T_{\mathbb{P}_n})$, the Leray–Hirsch theorem gives

$$\dim H^{2k}(X_0, \mathbb{Q}) = k + 1.$$

Now if we denote by H the class of a hyperplane in \mathbb{P}_n , and by L the class of an ample divisor on X_0 , then the classes

$$L^k, L^{k-1} \cdot (\phi_0^* H), \dots, L \cdot (\phi_0^* H)^{k-1}, (\phi_0^* H)^k \quad (3)$$

form a basis of $H^{2k}(X_0, \mathbb{Q})$, which can be seen as follows: By the dimension formula given above, it is sufficient to show linear independency, so assume that we are given $\lambda_0, \dots, \lambda_k \in \mathbb{Q}$ such that

$$\sum_{\ell=0}^k \lambda_\ell L^{k-\ell} \cdot (\phi_0^* H)^\ell = 0. \quad (4)$$

Now let $\ell_0 \in \{0, \dots, k\}$. By induction, we assume that $\lambda_\ell = 0$ for all $\ell < \ell_0$. Then intersecting (4) with $L^{n-k-1+\ell_0} \cdot (\phi_0^* H)^{n-\ell_0}$ yields

$$\lambda_{\ell_0} L^{n-1} \cdot (\phi_0^* H)^n = 0,$$

thus $\lambda_{\ell_0} = 0$ since $L^{n-1} \cdot (\phi_0^* H)^n > 0$.

So (3) is indeed a basis of $H^2(X_0, \mathbb{Q})$ and we can write

$$[F_0] = \sum_{\ell=0}^k \alpha_\ell L^{k-\ell} \cdot (\phi_0^* H)^\ell \quad (5)$$

for some $\alpha_0, \dots, \alpha_k \in \mathbb{Q}$. We now let $\ell_0 \in \{0, \dots, k\}$ and assume that $\alpha_\ell = 0$ for $\ell < \ell_0$. We observe that $[F_0] \cdot (\phi_0^* H)^{n-\ell_0} = 0$ since F_0 is contained in a fiber of ϕ_0 and $\ell_0 \leq k < n$. Hence, intersecting (5) with $L^{n-k-1+\ell_0} \cdot (\phi_0^* H)^{n-\ell_0}$ yields

$$0 = \alpha_{\ell_0} L^{n-1} \cdot (\phi_0^* H)^n,$$

so we deduce $\alpha_{\ell_0} = 0$ as before. Therefore by induction, we have $[F_0] = 0$, which is impossible, X_0 being projective. \square

Now we set $V = \mathcal{F}|_{X_0}$. If the bundle V is semi-stable, then $V \simeq T_{\mathbb{P}_n}$ and the theorem is settled. \square

Suppose in Theorem 5.1 that $X_0 \simeq \mathbb{P}(V)$ with an unstable bundle V (we will show in section 6 that this can indeed occur). Then X_0 does not carry a contact structure. In fact, otherwise $X_0 \simeq \mathbb{P}(T_S)$ with some projective variety S , [KPSW00]. Hence X_0 has two extremal contractions, and therefore X_0 is Fano. Hence T_S is ample and thus $S \simeq \mathbb{P}_n$ (or apply Wiśniewski's theorem). Therefore we may state the following

Corollary 5.5. *Let $\pi: \mathcal{X} \rightarrow \Delta$ be a family of compact manifolds. Assume $X_t \simeq \mathbb{P}(T_{\mathbb{P}_n})$ for $t \neq 0$. If X_0 is a projective contact manifold, then $X_0 \simeq \mathbb{P}(T_{\mathbb{P}_n})$.*

In the situation of Theorem 5.1, we had two contact structures on X_t . This phenomenon is quite unique because of the following result [KPSW00], Prop. 2.13.

Theorem 5.6. *Let X be a projective contact manifold of dimension $2n - 1$ admitting two extremal rays in the cone of curves $\overline{NE}(X)$. Then $X \simeq \mathbb{P}(T_{\mathbb{P}_n})$.*

Here is an extension of Theorem 5.6 to the non-algebraic case.

Theorem 5.7. *Let X be a compact contact Kähler manifold admitting two contractions $\phi_i: X \rightarrow Y_i$ to normal compact Kähler spaces Y_i . This is to say that $-K_X$ is ϕ_i -ample and that $\rho(X/Y_i) = 1$. Then X is projective and therefore $X = \mathbb{P}(T_{\mathbb{P}_n})$.*

Proof. We already know by Theorem 3.13 that $X = \mathbb{P}(T_{Y_1})$. Let $F \simeq \mathbb{P}_{n-1}$ be a fiber of ϕ_2 . Then the restriction $\phi_1|_F$ is finite. We claim that Y_1 must be projective. In fact, consider the rational quotient, say $f: Y_1 \dashrightarrow Z$, which is an almost holomorphic map to a compact Kähler manifold Z . By construction, the map f contracts the images $\phi_1(F)$, hence $\dim Z \leq 1$. But then Z is projective and therefore Y_1 is projective, too (e.g. by arguing that y_1 cannot carry a holomorphic 2-form).

By symmetry, Y_2 is projective, too. Since the morphisms ϕ_i induce a finite map $X \rightarrow Y_1 \times Y_2$ (onto the image of X), the variety X is also projective. \square

Any projective contact manifold X with $b_2(X) \geq 2$ is of the form $X = \mathbb{P}(T_Y)$. Therefore it is natural to ask for generalizations of Theorem 5.1, substituting the projective space by other projective varieties.

Proposition 5.8. *Let $\pi: \mathcal{X} \rightarrow \Delta$ be a projective family of compact manifolds X_t of dimension $2n - 1$. Assume that $X_t \simeq \mathbb{P}(T_{Y_t})$ for $t \neq 0$ with (necessarily projective) manifolds $Y_t \neq \mathbb{P}_n$. Assume that $H^q(X_t, \mathcal{O}_{X_t}) = 0$ for $q = 1, 2$ for some (hence all) t . Then the following statements hold.*

- (1) *There exists a relative contraction $\Phi: \mathcal{X} \rightarrow \mathcal{Y}$ over Δ such that $\Phi|_{X_t}$ is the given \mathbb{P}_{n-1} -bundle structure for $t \neq 0$.*
- (2) *If $\phi_0 := \Phi|_{X_0}$ is equidimensional, then $X_0 \simeq \mathbb{P}(\mathcal{E}_0)$ with a rank- n bundle \mathcal{E} over the projective manifold Y_0 ; and Y_0 is the limit manifold of a family $\tau: \mathcal{Y} \rightarrow \Delta$ such that $Y_t \simeq \tau^{-1}(t)$ for $t \neq 0$. In other words, $\mathcal{X} \simeq \mathbb{P}(\mathcal{E})$ such that $\mathcal{E} = T_{\mathcal{Y}/\Delta}$ over $\Delta \setminus \{0\}$.*

Proof. Since $Y_t \neq \mathbb{P}_n$ by assumption, every X_t , $t \neq 0$, has a unique Mori contraction, the projection $\psi_t: X_t \rightarrow Y_t$, by Theorem 5.6. As in the proof of Theorem 5.1, we obtain a relative Mori contraction

$$\Phi: \mathcal{X} \rightarrow \mathcal{Y}$$

over Δ , and necessarily $\Phi|_{X_t} = \phi_t$ for all $t \neq 0$ (we use again [Wi91b], (1.3)). This already shows Claim (1).

If ϕ_0 is equidimensional, we apply—as in the proof of Theorem 5.1—[BS95], (III.3.2.1), to conclude that there exists a locally free sheaf \mathcal{E}_0 of rank n on Y_0 such that $X_0 \simeq \mathbb{P}(\mathcal{E}_0)$, proving (2). \square

Theorem 5.9. *Let $\pi: \mathcal{X} \rightarrow \Delta$ be a projective family of compact manifolds X_t of dimension $2n - 1$. Assume that $X_t \simeq \mathbb{P}(T_{Y_t})$ for $t \neq 0$ with (necessarily projective) manifolds $Y_t (\neq \mathbb{P}_n)$. Assume that $H^q(X_t, \mathcal{O}_{X_t}) = 0$ for $q = 1, 2$ for some (hence all) t . Assume moreover that*

- (1) $\dim X_0 \leq 5$, or
- (2) $b_{2j}(Y_t) = 1$ for some $t \neq 0$ and all $1 \leq j < \frac{n}{2}$.

Then there exists a relative contraction $\Phi: \mathcal{X} \rightarrow \mathcal{Y}$ over Δ such that $\Phi|_{X_t}$ is the given \mathbb{P}_{n-1} -bundle structure for $t \neq 0$. Moreover there is a locally free sheaf \mathcal{E} on \mathcal{Y} such that $\mathcal{X} \simeq \mathbb{P}(\mathcal{E})$ and $\mathcal{E}|_{Y_t} \simeq T_{Y-t}$ for all $t \neq 0$.

Proof. By the previous proposition it suffices to show that $\phi_0 = \Phi|_{X_0}$ is equidimensional.

- (1) First suppose that $\dim X_0 \leq 5$. Then $1 \leq \dim Y_0 \leq 3$. The case $\dim Y_0 = 1$ is trivial. If $\dim Y_0 = 2$, then all fibers must have codimension 2, because ϕ_0 does not contract a divisor (the relative Picard number being 1). If $\dim Y_0 = 3$, then by [AW97], (5.1), we cannot have a 3-dimensional fiber. Since again there is no 4-dimensional fiber, ϕ_0 must be equidimensional also in this case.

(2) If $b_{2j}(Y_t) = 1$ for some t and all $1 \leq j \leq \frac{n}{2}$, then $b_{2k}(X_t) = k + 1$ for $k < n$ and we may simply argue as in Sublemma 5.4 to conclude that ϕ_0 is equidimensional (the smoothness of Y_0 is not essential in the reasoning of Sublemma 5.4). \square

6. DEGENERATIONS OF $T_{\mathbb{P}_n}$

In view of Theorem 5.1, we can ask the question which bundles can occur as degenerations of $T_{\mathbb{P}_n}$, i.e., for which rank- n bundles V on \mathbb{P}_n there exists a rank- n bundle \mathcal{V} on $\mathbb{P}_n \times \Delta$ such that

$$\mathcal{V}_t := \mathcal{V}|_{\mathbb{P}_n \times \{t\}} \simeq \begin{cases} T_{\mathbb{P}_n}, & \text{for } t \neq 0, \\ V, & \text{for } t = 0. \end{cases}$$

In the case that $n \geq 3$ is odd, it was already observed by Hwang in [Hw06] that one can easily construct a nontrivial degeneration of $T_{\mathbb{P}_n}$ as follows: We consider the null correlation bundle on \mathbb{P}_n , which is a rank- $(n-1)$ bundle N on \mathbb{P}_n given by a short exact sequence

$$0 \longrightarrow N \longrightarrow T_{\mathbb{P}_n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}_n}(1) \longrightarrow 0.$$

(cf. [OSS80], (I.4.2)). The existence of this sequence now implies that $T_{\mathbb{P}_n}$ can be degenerated to $N(1) \oplus \mathcal{O}_{\mathbb{P}_n}(2)$.

When n is even, matters become more complicated, but we can still obtain nontrivial degenerations:

Proposition 6.1. *Let $n \geq 2$. Then there exists a rank- n bundle \mathcal{V} on $\mathbb{P}_n \times \Delta$ such that $\mathcal{V}_t \simeq T_{\mathbb{P}_n}$ for $t \neq 0$ and $h^0(\mathcal{V}_0(-2)) = 1$, so in particular $\mathcal{V}_0 \not\simeq T_{\mathbb{P}_n}$.*

Proof. We construct an inclusion of vector bundles

$$A: \Omega_{\mathbb{P}_n \times \Delta / \Delta}^1(2) \oplus \mathcal{O}_{\mathbb{P}_n \times \Delta} \hookrightarrow \mathcal{O}_{\mathbb{P}_n \times \Delta}(1)^{\oplus(n+1)} \oplus \Omega_{\mathbb{P}_n \times \Delta / \Delta}^1(2)$$

via a family $A = (A_t)_{t \in \Delta}$ of matrices

$$A_t = \begin{pmatrix} \alpha_t & \beta_t \\ \gamma_t & \delta_t \end{pmatrix}$$

of sheaf homomorphisms

$$\begin{aligned} \alpha_t: \Omega_{\mathbb{P}_n}^1(2) &\rightarrow \mathcal{O}_{\mathbb{P}_n}(1)^{\oplus(n+1)}, & \beta_t: \mathcal{O}_{\mathbb{P}_n} &\rightarrow \mathcal{O}_{\mathbb{P}_n}(1)^{\oplus(n+1)}, \\ \gamma_t: \Omega_{\mathbb{P}_n}^1(2) &\rightarrow \Omega_{\mathbb{P}_n}^1(2), & \delta_t: \mathcal{O}_{\mathbb{P}_n} &\rightarrow \Omega_{\mathbb{P}_n}^1(2), \end{aligned}$$

which we define as follows: We take α_t and β_t to be the natural inclusions coming from the Euler sequence and its dual, where we choose the coordinates on \mathbb{P}_n such that

$$\beta_t(\mathcal{O}_{\mathbb{P}_n}) \not\subset \alpha_t(\Omega_{\mathbb{P}_n}^1(2)).$$

This implies that the map

$$\alpha_t \oplus \beta_t: \Omega_{\mathbb{P}_n}^1(2) \oplus \mathcal{O}_{\mathbb{P}_n} \rightarrow \mathcal{O}_{\mathbb{P}_n}(1)^{\oplus(n+1)}$$

is generically surjective. Since $\Omega_{\mathbb{P}_n}^1(2) \simeq \Lambda^{n-1}(T_{\mathbb{P}_n}(-1))$ is globally generated, a general section in $H^0(\Omega_{\mathbb{P}_n}^1(2))$ has only finitely many zeroes. Since $\Omega_{\mathbb{P}_n}^1(2)$ is homogeneous, we can thus choose the map δ_t in such a way that its zeroes are disjoint from the locus where $\alpha_t \oplus \beta_t$ is not surjective. Finally we let $\gamma_t = t \cdot \text{id}$.

Now in order to show that A is an inclusion of vector bundles, we need to show that for any point $(p, t) \in \mathbb{P}_n \times \Delta$, the matrix

$$A_t(p) = \begin{pmatrix} \alpha_t(p) & \beta_t(p) \\ \gamma_t(p) & \delta_t(p) \end{pmatrix} \in \mathbb{C}^{(2n+1) \times (n+1)}$$

has rank $n + 1$. For semicontinuity reasons, shrinking Δ if necessary, we can assume $t = 0$, then the rank condition follows easily from the choice of $\alpha_0, \beta_0, \gamma_0, \delta_0$.

We now let

$$\mathcal{V} := \text{coker } A.$$

It remains to investigate the properties of the bundles $\mathcal{V}_t := \mathcal{V}|_{\mathbb{P}_n \times \{t\}}$. For each $t \in \Delta$, we have an exact sequence of vector bundles

$$0 \longrightarrow \Omega_{\mathbb{P}_n}^1(2) \oplus \mathcal{O}_{\mathbb{P}_n} \longrightarrow \mathcal{O}_{\mathbb{P}_n}(1)^{\oplus(n+1)} \oplus \Omega_{\mathbb{P}_n}^1(2) \longrightarrow \mathcal{V}_t \longrightarrow 0. \quad (6)$$

We want to calculate $H^q(\mathcal{V}_t(-1-k))$ for $k = 0, \dots, n$. From the Bott formula we obtain for $(k, q) \in \{0, \dots, n\}^2$:

$$h^q(\Omega_{\mathbb{P}_n}^1(1-k)) = \begin{cases} 1, & \text{for } (k, q) = (1, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Now if we tensorize (6) with $\mathcal{O}_{\mathbb{P}_n}(-1-k)$, take the long exact cohomology sequence and observe that $H^q(\delta_0) = 0$ for every q , we get for $(k, q) \in \{0, \dots, n\}^2$:

$$h^q(\mathcal{V}_0(-1-k)) = \begin{cases} n+1, & \text{for } (k, q) = (0, 0), \\ 1, & \text{for } (k, q) \in \{(1, 0), (1, 1), (n, n-1)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, if we observe that $H^q(\delta_t) = \text{id}$ for $t \neq 0$, we obtain for $t \neq 0$, $(k, q) \in \{0, \dots, n\}^2$:

$$h^q(\mathcal{V}_0(-1-k)) = \begin{cases} n+1, & \text{for } (k, q) = (0, 0), \\ 1, & \text{for } (k, q) \in \{(n, n-1)\}, \\ 0, & \text{otherwise.} \end{cases}$$

The proposition now follows from Lemma 6.2. \square

Lemma 6.2. *Let V be a vector bundle on \mathbb{P}_n such that for any $(k, q) \in \{0, \dots, n\}^2$, we have*

$$h^q(V(-1-k)) = \begin{cases} n+1, & \text{for } (k, q) = (0, 0), \\ 1, & \text{for } (k, q) = (n, n-1), \\ 0, & \text{otherwise.} \end{cases}$$

Then $V \simeq T_{\mathbb{P}_n}$.

Proof. We consider the Beilinson spectral sequence for the bundle $V(-1)$, which has E_1 -term

$$E_1^{pq} = H^q(V(-1+p)) \otimes \Omega_{\mathbb{P}_n}^{-p}(-p)$$

(cf. [OSS80], (II.3.1.3)).

By assumption, $E_1^{pq} = 0$ for $(p, q) \notin \{(0, 0), (-n, n-1)\}$ and

$$E_1^{0,0} = \mathcal{O}_{\mathbb{P}_n}^{\oplus(n+1)}, \quad E_1^{-n,n-1} = \mathcal{O}_{\mathbb{P}_n}(-1).$$

In particular, the only nonzero differential occurs at the E_n -term, namely

$$d_n^{-n,n-1}: E_n^{-n,n-1} \rightarrow E_n^{0,0}.$$

Since $E_\infty^{pq} = 0$ for $p+q \neq 0$ and $E_\infty^{-p,p}$ are the quotients of a filtration of $V(-1)$, the differential $d_n^{-n,n-1}$ induces a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_n}(-1) \xrightarrow{d_n^{-n,n-1}} \mathcal{O}_{\mathbb{P}_n}^{\oplus(n+1)} \longrightarrow V(-1) \longrightarrow 0. \quad (7)$$

Now since V is locally free, the map $d_n^{-n,n-1}$ cannot have zeroes, so (7) must be an Euler sequence, whence $V(-1) \simeq T_{\mathbb{P}_n}(-1)$. \square

7. DEFORMATIONS II: POSITIVE IRREGULARITY

A homogeneous compact contact Kähler manifold X of dimension $2n+1$ with $b_2(X) \geq 2$ is either $\mathbb{P}(T_{\mathbb{P}_{n+1}})$ or a product $A \times \mathbb{P}_n$ with a torus A of dimension $n+1$. Here we study in general the Kähler deformations of $A \times \mathbb{P}_n$, where A is an m -dimensional torus.

Theorem 7.1. *Let $\pi: \mathcal{X} \rightarrow \Delta$ be a family of compact manifolds over the unit disc $\Delta \subset \mathbb{C}$. Assume $X_t \simeq A_t \times \mathbb{P}_n$ for $t \neq 0$, where A_t is a torus of dimension m . If X_0 is in class \mathcal{C} , then the relative Albanese morphism realises \mathcal{X} as a submersion $\alpha: \mathcal{X} \rightarrow \mathcal{A}$, where $\pi: \mathcal{A} \rightarrow \Delta$ is torus bundle such that $\pi^{-1}(t) \simeq A_t$ for $t \neq 0$. Moreover there is a locally free sheaf \mathcal{E} over \mathcal{A} such that $\mathcal{X} = \mathbb{P}(\mathcal{E})$, $\mathcal{X}_t \simeq \mathbb{P}(\mathcal{E}_t)$ for all t and $\mathcal{E}|_{A_t} \simeq \mathcal{O}_{A_t}^{n+1}$ for $t \neq 0$.*

Proof. Let $m = \dim A_t = q(X_t)$ for $t \neq 0$. Hodge decomposition on X_0 gives $q(X_0) = m$. Let

$$\alpha: \mathcal{X} \rightarrow \mathcal{A}$$

be the relative Albanese map. Then $\mathcal{A} \rightarrow \Delta$ is a torus bundle and

$$\alpha_t = \alpha|_{X_t}: X_t \rightarrow A_t$$

is the Albanese map for all t . Since α_t is surjective for all $t \neq 0$, the map α is surjective, too, and so is α_0 . We may choose relative vector fields

$$v_1, \dots, v_m \in H^0(\mathcal{X}, T_{\mathcal{X}/\Delta}),$$

such that for all t , the push-forwards $(\alpha_t)_*(v_i|_{X_t})$ form a basis of $H^0(A_t, T_{A_t})$. Since singular fibers are mapped to singular fibers by automorphisms of X_0 , it follows that the singular locus S of α_0 , i.e., the set of points $a \in A_0$ such that the fiber over a is singular, is invariant in A_0 under the automorphism group. Hence $S = \emptyset$, so that α_0 is a submersion like all the other maps α_t . Therefore α is a bundle, with fibers \mathbb{P}_n over Δ^* . The global rigidity of the projective space [Si89] applied to local sections of \mathcal{A} over Δ , passing through A_0 , implies that all fibers of α are \mathbb{P}_n . The existence of \mathcal{E} follows from [El82], (4.3). \square

Remark 7.2. Popovici [Po09] has shown that any global deformation of projective manifolds is in class \mathcal{C} , so that the assumption in Theorem 6.1 that X_0 is in class \mathcal{C} can be removed in case X_t is projective. The Kähler version of Popovici's theorem is still open.

Example 7.3. We cannot conclude in Theorem 6.1 that $X_0 \simeq A_0 \times \mathbb{P}_n$, even if $m = n = 1$. Take e.g. a rank-2 vector bundle \mathcal{F} over $\mathbb{P}_1 \times \Delta$ such that $\mathcal{F}|_{\mathbb{P}_1 \times \{t\}} = \mathcal{O}^2$ for $t \neq 0$ and $\mathcal{F}|_{\mathbb{P}_1 \times \{0\}} = \mathcal{O}(1) \oplus \mathcal{O}(-1)$. Let $\eta: A \rightarrow \mathbb{P}_1$ be a two-sheeted covering from an elliptic curve A and set $\mathcal{E} = (\eta \times \text{id})^*(\mathcal{F})$. Then $\mathcal{X} = \mathbb{P}(\mathcal{E})$ is a family of compact surfaces X_t such that $X_t = A \times \mathbb{P}_1$ for $t \neq 0$ but X_0 is not a product. Notice also that X_0 is not almost homogeneous.

It is a trivial matter to modify this example to obtain a map to a 2-dimensional torus which is a product of elliptic curves. Therefore the limit of a Kähler contact manifold with positive irregularity might not be a contact manifold again.

Corollary 7.4. *Assume the situation of Theorem 6.1. Then the following assertions are equivalent.*

- (1) $X_0 \simeq A_0 \times \mathbb{P}_n$.
- (2) \mathcal{E}_0 is semi-stable for some Kähler class ω .
- (3) X_0 is homogeneous.
- (4) X_0 is almost homogeneous.

Proof. (1) implies (2). Under the assumption of (1), there is a line bundle L on A_0 such that $\mathcal{E}_0 \simeq L^{\oplus n+1}$. Hence \mathcal{E} is semi-stable for actually any choice of ω .

(2) implies (3). From the semi-stability of \mathcal{E}_0 and $h^0(\mathcal{E}_0) \geq n+1$, it follows easily that \mathcal{E}_0 is trivial and that X_0 is homogeneous as product $A_0 \times \mathbb{P}_n$. In fact, choose $n+1$ sections of \mathcal{E}_0 and consider the induced map $\mu: \mathcal{O}_{A_0}^{n+1} \rightarrow \mathcal{E}_0$. By the stability of \mathcal{E}_0 , the map μ is generically surjective. Hence $\det \mu \neq 0$, hence an isomorphism, so that μ itself is an isomorphism.

The implication “(3) implies (4)” is obvious.

(4) implies (1). Consider the tangent bundle sequence

$$0 \rightarrow T_{X_0/A_0} \rightarrow T_{X_0} \rightarrow \alpha_0^*(T_{A_0}) \rightarrow 0.$$

Since X_0 is almost homogeneous, all vector fields on A_0 must lift to X_0 . Consequently the connecting map

$$H^0(X_0, \pi^*(T_{A_0})) \rightarrow H^1(X_0, T_{X_0/A_0})$$

vanishes, and therefore the tangent bundle sequence splits. Let $\mathcal{F} = \alpha_0^*(T_{A_0})$. Then $\mathcal{F} \subset T_{X_0}$ is clearly a foliation and it has compact leaves (the limits of tori in $A_t \times \mathbb{P}_n$). By [Hoe07], 2.4.3, there exists an equi-dimensional holomorphic map $f: X_0 \rightarrow Z_0$ to a compact variety Z_0 such that the set-theoretical fibers F of f are leaves of \mathcal{F} . Since the fibers F have an étale map to A_0 , they must be tori again. It is now immediate that $Z_0 = \mathbb{P}_n$ and that $X_0 = A_0 \times \mathbb{P}_n$. \square

Corollary 7.5. *Assume in Theorem 6.1 that $m = 2$ and $n = 1$. Then either $X_0 \simeq A_0 \times \mathbb{P}_1$, or $X_0 = \mathbb{P}(\mathcal{E}_0)$ and one of the following holds:*

- (1) *There is a torus bundle $p: A_0 \rightarrow B_0$ to an elliptic curve B_0 and the rank-2 bundle \mathcal{E}_0 on A_0 sits in an extension*

$$0 \rightarrow p^*(\mathcal{L}_0) \rightarrow \mathcal{E}_0 \rightarrow p^*(\mathcal{L}_0^*) \rightarrow 0$$

with $\deg \mathcal{L}_0 > 0$.

- (2) *The rank 2-bundle \mathcal{E}_0 sits in an extension*

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{E}_0 \rightarrow \mathcal{I}_Z \otimes \mathcal{S}^* \rightarrow 0$$

with an ample line bundle \mathcal{S} and a finite non-empty set Z of degree $\deg Z = c_1(\mathcal{S})^2$.

Proof. By Corollary 6.4 we may assume that \mathcal{E}_0 is not semi-stable for some (or any) Kähler class ω . Let \mathcal{S} be a maximal destabilising subsheaf, which is actually a line bundle, leading to an exact sequence

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{E}_0 \rightarrow \mathcal{I}_Z \otimes \mathcal{S}^* \rightarrow 0.$$

Here Z is a finite set or empty. Taking c_2 and observing that $c_2(\mathcal{E}_0) = 0$ gives

$$c_1(\mathcal{S})^2 = \deg Z.$$

The destabilisation property reads

$$c_1(\mathcal{S}) \cdot \omega > 0.$$

Since $h^0(\mathcal{E}_0) \geq 2$, we deduce that $h^0(\mathcal{S}) \geq 2$, in particular, \mathcal{S} is nef, \mathcal{S} being maximal destabilizing.

If \mathcal{S} is ample, there is nothing more to prove, hence we may assume that \mathcal{S} is not ample. \mathcal{S} being nef, $c_1(\mathcal{S})^2 = 0$ and \mathcal{S} defines a submersion $p: A_0 \rightarrow$

B_0 to an elliptic curve B_0 such that there exists an ample line bundle \mathcal{L}_0 with $\mathcal{S} = p^*(\mathcal{L}_0)$. Therefore we obtain an extension

$$0 \rightarrow p^*(\mathcal{L}_0) \rightarrow \mathcal{E}_0 \rightarrow p^*(\mathcal{L}_0^*) \rightarrow 0,$$

as required. \square

Remark 7.6. The second case in Corollary 7.5 really occurs. Take a finite map $f: \mathcal{A} \rightarrow \mathbb{P}_2 \times \Delta$ over Δ and a rank-2 bundle \mathcal{F} on $\mathbb{P}_2 \times \Delta$ such that $\mathcal{F}|_{\mathbb{P}_2 \times \{t\}} \simeq \mathcal{O}^2$ for $t \neq 0$ and such that \mathcal{F}_0 is not trivial. For examples see e.g. [Sc83]. Now $\mathcal{E} = f^*(\mathcal{F})$ gives an example we are looking for.

Corollary 7.7. *Assume in Theorem 6.1 that $m = 2$ and $n = 1$. Let $\Phi: T_{\mathcal{X}/\Delta} \rightarrow \frac{-K_{\mathcal{X}}}{2}$ be a morphism such that $\Phi|_{X_t} = \phi_t$ is a contact morphism (i.e., defines a contact structure) for $t \neq 0$. Suppose that*

$$\phi_0: T_{X_0} \rightarrow \frac{-K_{X_0}}{2}$$

does not vanish identically. Then the kernel \mathcal{F}_0 of ϕ_0 is integrable (in contrast to the maximally non-integrable bundle \mathcal{F}_t).

Proof. We consider a family (ϕ_t) of morphisms

$$\phi_t: T_{X_t} \rightarrow \mathcal{H}_t$$

such that ϕ_t is a contact form for $t \neq 0$ and $-K_{X_t} = 2\mathcal{H}_t$. Consider the (torsion free) kernel \mathcal{F}_0 of ϕ_0 . We need to show that the induced map

$$\mu: \left(\bigwedge^2 \mathcal{F}_0\right)^{**} = \det \mathcal{F}_0 \rightarrow \mathcal{H}_0.$$

vanishes. Since the determinant of the kernel \mathcal{F}_t of ϕ_t is isomorphic to \mathcal{H}_t , we conclude that

$$\det \mathcal{F}_0 \simeq \mathcal{H}_0 \otimes \mathcal{O}_{X_0}(E) \tag{*}$$

with an effective (possibly vanishing) divisor E on X_0 . Now the induced map

$$\mu: \det \mathcal{F}_0 \rightarrow \mathcal{H}_0$$

must have zeroes, otherwise X_0 would be a contact manifold, hence $X_0 \simeq A_0 \times \mathbb{P}_1$. Thus $\mu = 0$ by (*), and \mathcal{F}_0 is integrable. \square

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